Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 220 (2008) 17–21
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Letter to the Editor

On the global convergence of Chebyshev's iterative method ☆

S. Amat[a], S. Busquier[a,*], J.M. Gutiérrez[b], M.A. Hernández[b]

[a] Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Spain
[b] Departamento de Matemáticas y Computación, Universidad de La Rioja, Spain

Received 13 March 2007

Abstract

In [A. Melman, Geometry and convergence of Euler's and Halley's methods, SIAM Rev. 39(4) (1997) 728–735] the geometry and global convergence of Euler's and Halley's methods was studied. Now we complete Melman's paper by considering other classical third-order method: Chebyshev's method. By using the geometric interpretation of this method a global convergence theorem is performed. A comparison of the different hypothesis of convergence is also presented.

MSC: 65F15; 65H05

Keywords: Nonlinear equations; Iterative methods; Geometry global convergence

1. Introduction

We consider the problem of finding the zeros of a given function f, that is, finding the values x^* for which $f(x^*)=0$. To approximate these equations we can use iterative methods. One of the most studied is Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

The geometric interpretation of Newton's method is well known, given an iterate x_n, the next iterate is the zero of the tangent line

$$y(x) - f(x_n) = f'(x_n)(x - x_n),$$

to the graph of f at $(x_n, f(x_n))$.

Newton's iteration can be considered as a fixed-point iteration

$$x_{n+1} = \Phi_N(x_n), \quad \Phi_N(x) = x - \frac{f(x)}{f'(x)}.$$

☆ The research of the first two authors has been supported by the Spanish Grants MTM2004-07114 and 00675/PI/04 and the research of the last two authors has been supported by grants of the Spanish Ministry of Education and Science (Ref. MTM 2005-03091).
* Corresponding author.
E-mail addresses: sergio.amat@upct.es (S. Amat), sonia.busquier@upct.es (S. Busquier), jmguti@dmc.unirioja.es (J.M. Gutiérrez), mahernan@dmc.unirioja.es (M.A. Hernández).

doi:10.1016/j.cam.2007.07.022

In general, an iterative method $x_{n+1}=\Phi(x_n)$ is of pth-order if the solution x^* satisfies $x^*=\Phi(x^*)$, $\Phi'(x^*)=\cdots=\Phi^{(p-1)}(x^*)=0$ and $\Phi^{(p)}(x^*)\neq 0$. For such a method, the error $|x^*-x_{n+1}|$ is proportional to $|x^*-x_n|^p$ as $n\to\infty$. It can be shown that the number of significant digits is multiplied by the order of convergence (approximately) by proceeding from x_n to x_{n+1}. For instance, Newton's method has quadratic convergence (order two) for simple roots.

The following well-known theorem, giving enough conditions for the global convergence of Newton's method, follows easily from its geometric interpretation.

Theorem 1. *Let f'' be continuous on an interval J, containing a root x^* of f; let $f'\neq 0$ and $f''\geqslant 0$ or $f''\leqslant 0$ on J. Then Newton's method converges monotonically to x^* from any point $x_0\in J$ such that $f(x_0)f''(x_0)\geqslant 0$.*

In [4] Melman analyzes the geometry and global convergence of two classical third-order iterative schemes: Euler's and Halley's methods. These methods are obtained, respectively, by considering a parabola and an hyperbola in the following way:

$$y(x)=ax^2+bx+c$$

and

$$axy(x)+y(x)+bx+c=0.$$

Then if we impose the following tangency conditions at a point x_n:

$$y(x_n)=f(x_n),\quad y'(x_n)=f'(x_n)\quad\text{and}\quad y''(x_n)=f''(x_n), \tag{2}$$

we have

$$y(x)=\frac{f''(x_n)}{2}(x-x_n)^2+f'(x_n)(x-x_n)+f(x_n)$$

and

$$y(x)-f(x_n)-f'(x_n)(x-x_n)-\frac{f''(x_n)}{2f'(x_n)}(x-x_n)(y(x)-f(x_n))=0,$$

respectively.

The next iterate x_{n+1} in both methods is obtained by calculating the intersection of the tangent curve with the OX-axis. The corresponding iterative schemes are the following sequences[1] :

$$x_{n+1}=x_n-\frac{1-\sqrt{1-2L_f(x_n)}}{f''(x_n)/f'(x_n)},\quad n\geqslant 0$$

and

$$x_{n+1}=x_n-\frac{2}{2-L_f(x_n)}\frac{f(x_n)}{f'(x_n)},\quad n\geqslant 0,$$

that is, the well-known *Euler's* and *Halley's methods*, respectively (see [4,5]).

In [4] we can find the following global convergence results for Euler's and Halley's methods:

Theorem 2. *Let f''' be continuous, $f'\neq 0$, and $f'f'''\leqslant 0$ on an interval J containing a root x^* of f. Then Euler's method converges monotonically to x^* from any point of the interval.*

Theorem 3. *Let f''' be continuous, $f'\neq 0$, and $((\eta f')^{-1/2})''\geqslant 0$ on an interval J containing a root x^* of f with $\eta=\operatorname{sgn}(f')$. Then Halley's method converges monotonically to x^* from any point of the interval.*

[1] Throughout this paper we denote

$$L_f(x)=\frac{f(x)f''(x)}{f'(x)^2}.$$

The aim of this paper is to complete Melman's study by considering other classical third-order method: Chebyshev's method.

In what follows, we consider Chebyshev's method. As we shall see later on, this method has similar geometric interpretation as before. With Halley's scheme, they are the most studied third-order methods in the literature. We derive sufficient conditions for its global convergence by using its geometric interpretation and compare the different hypothesis.

The paper consist on two parts: in the first one we present the geometric interpretation of the mentioned iterative method and we analyze its global convergence. In the second one, some comparisons are discussed.

2. Geometry and convergence of Chebyshev's method

Chebyshev's method is obtained by quadratic interpolation of the inverse function of f, in order to approximate $f^{-1}(0)$ [6]. But it also admits a geometric derivation, from a parabola in the form

$$ay(x)^2 + y(x) + bx + c = 0, \tag{3}$$

that after the imposition of the tangency conditions (2) can be written

$$-\frac{f''(x_n)}{2f'(x_n)^2}(y(x) - f(x_n))^2 + y(x) - f(x_n) - f'(x_n)(x - x_n) = 0.$$

By calculating the intersection of this parabola with the OX-axis we obtain the next step of Chebyshev's method

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}L_f(x_n)\right)\frac{f(x_n)}{f'(x_n)}.$$

We refer to [1] for the geometric interpretation of other third-order methods.

By using the geometric interpretation of Chebyshev's method, we can obtain the following global convergence theorem.

Theorem 4. *Let f''' be continuous on an interval J containing a root x^* of f, let $f' \neq 0$, $L_f(x) > -2$ and $((\eta/f'(x))^2)'' \geqslant 0$ in J, with $\eta = \operatorname{sgn}(f')$. Then Chebyshev's method converges monotonically to x^* from any point of the interval.*

Proof. We suppose $f' > 0$ (for $f' < 0$ the proof is similar).

First, we begin from a point on the left of x^*, $\overline{x} \leqslant x^*$.

We would like to show that the intersection $\widehat{x}$ of the parabola $y(x)$ given in (3) with the OX-axis will be in $[\overline{x}, x^*]$. By hypothesis $L_f(\overline{x}) > -2$, in particular, $\overline{x} \leqslant \widehat{x}$.

Thus, it will be enough, if for $x \geqslant \overline{x}$, we can proof that

$$y(x) = \frac{-1 + \sqrt{1 - 4a(bx + c)}}{2a} \geqslant f(x). \tag{4}$$

In this case, we will obtain a monotonic increasing sequence, bounded from above by x^*, then it converges at the limit $\gamma \leqslant x^*$. So, because the construction of the method and the continuity of f the convergence is obtained, $\gamma = x^*$.

Inequality (4) is equivalent to

$$\frac{-1 + \sqrt{1 - 4a(bx + c)}}{2a} - \frac{-1 + \sqrt{1 - 4a(b\overline{x} + c)}}{2a} \geqslant f(x) - f(\overline{x})$$

or

$$\int_{\overline{x}}^{x} \frac{-b}{\sqrt{1 - 4a(bt + c)}}\,\mathrm{d}t \geqslant \int_{\overline{x}}^{x} f'(t)\,\mathrm{d}t. \tag{5}$$

As $f' > 0$ then for hypothesis $((1/f')^2)'' \geqslant 0$ in J, i.e., $(1/f')^2$ is convex, and therefore

$$\left(\frac{1}{f'(x)}\right)^2 \geqslant \frac{1 - 4a(bx + c)}{(-b)^2},$$

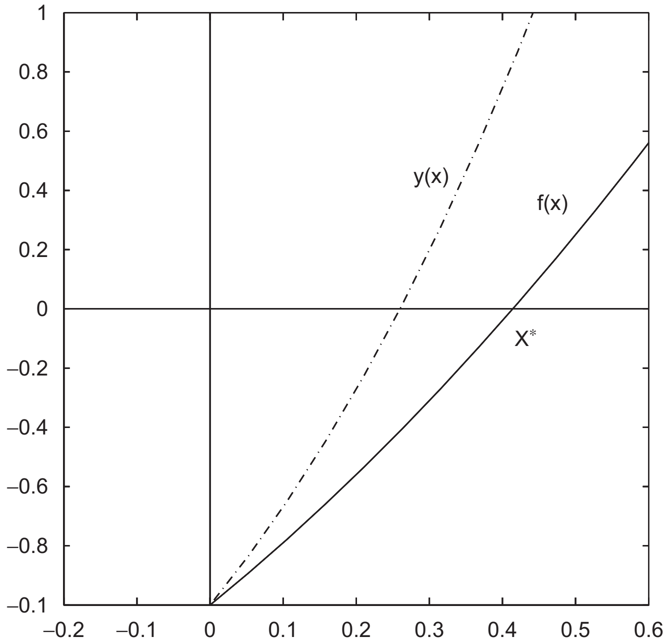

Fig. 1. "$f'(x) > 0,\ f(x) \leqslant y(x),\ x \in [\overline{x}, x^*]$."

because $(1 - 4a(bx + c))/(-b)^2$ approximates $(1/f'(x))^2$ up to second order.

Thus,

$$\frac{-b}{\sqrt{1 - 4a(bx + c)}} \geqslant f'(x) > 0$$

and consequently the relation (5) holds.

Finally, if we begin from a point on the right of the root, we will obtain $(-1 + \sqrt{1 - 4a(bx + c)})/2a \leqslant f(x)$ and the convergence will be monotonic from the right (Fig. 1). $\square$

3. A comparison of the sufficient conditions for global convergence

In the situation where the sufficient conditions of all theorems hold, Newton's method will always produce a smaller step than third-order methods. The reason for this is that when f is convex (concave), the tangent to the function at the approximation point always lie below (above) those third-order approximations (see [4]).

In the Theorem 4, in comparison with Theorems 2 and 3,, we can see the "extra" condition $L_f(x) > -2$. This, hypothesis is necessary as we can see in the following example:

Example 1. Let us consider $f(x) = x^2 - 1$, $J = (0, 2)$, $x^* = 1$ and $\overline{x} = \frac{1}{4}$.

In this case, $f'(x) = 2x > 0$, $f'''(x) = 0$, $\mu = 1$, $(1/(f'(x))^2)'' = 3/2x^4 > 0$, but the intersection of the parabola $y(x)$ given in (3) with the OX-axis is

$$\widehat{x} = \tfrac{1}{4} - 11\tfrac{15}{32} = -4.90625 \notin J.$$

On the other hand, "a priori" we need similar conditions for Euler's and Halley's methods. Indeed, that $L_f(x) \leqslant \frac{1}{2}$ and $L_f(x) \leqslant 2$, respectively. Nevertheless, these conditions are consequence of the rest of conditions appearing in the theorems and the particular forms of these methods. We analyze the situation for Halley's method (similar arguments work for Euler's method).

Let f''' be continuous, $f' \neq 0$ on an interval J containing a root x^* of f. We suppose $f' > 0$ (for $f' < 0$ the proof is similar).

We begin from a point on the left of x^*, $\overline{x} \leqslant x^*$ (for a point on the right, the proof is similar). Thus, $f(\overline{x}) < 0$.

Let $y(x) = a + b/(x + c)$ the hyperbola used in Halley's method. We would like to check if it is possible that the intersection $\widehat{x}$ of the hyperbola $y(x)$ with the OX-axis can be on the left of $\overline{x}$.

From the definition of Halley's method, $\widehat{x} - \overline{x} < 0$ if and only if

$$L_f(\overline{x}) = \frac{f(\overline{x})f''(\overline{x})}{f'(\overline{x})^2} > 2, \tag{6}$$

in particular, it is necessary that $f(\overline{x})f''(\overline{x}) \geqslant 0$, thus $f''(\overline{x}) \leqslant 0$.

By definition of $\overline{x}$ and $\widehat{x}$, we have $y(\widehat{x}) = 0$, $f(\overline{x}) = y(\overline{x})$, $f'(\overline{x}) = y'(\overline{x})$ and $f''(\overline{x}) = y''(\overline{x})$.

Using a Taylor's expansion around $\overline{x}$, we obtain

$$0 = y(\widehat{x}) = y(\overline{x}) + y'(\overline{x})(\widehat{x} - \overline{x}) + \frac{y''(\overline{x})}{2}(\widehat{x} - \overline{x})^2 + \frac{y'''(\psi)}{6}(\widehat{x} - \overline{x})^3. \tag{7}$$

Since $y'''(x) = -6b/(x + c)^4$ and $y'(x) = -b/(x + c)^2$, then $y'''(x) > 0$ as $y'(x)$. Moreover, $\widehat{x} - \overline{x} < 0$, $y(\overline{x}) < 0$, $y'(\overline{x}) > 0$ and $y''(\overline{x}) < 0$, in particular, the right part of Eq. (7) is less than zero and this is a contradiction. Therefore, it is not possible that

$$\frac{f(\overline{x})f''(\overline{x})}{f'(\overline{x})^2} > 2.$$

In certain cases, it is possible to compare the third-order methods. Considering, for instance, the case $f'(x) > 0$ and that $L_f(x) > -2$, the conditions of the Theorems 3 and 4 are equivalent to

$$\tfrac{3}{2} f''(x)^2 \geqslant f'(x) f'''(x),$$
$$3 f''(x)^2 \geqslant f'(x) f'''(x),$$

respectively. The condition of the Theorem 2 is $f'''(x) \leqslant 0$, that is, the strongest one.

For the Theorems 3 and 4 the convergence conditions could also have been written in terms of the function

$$L_{f'}(x) = \frac{f'(x)f'''(x)}{f''(x)^2}.$$

Indeed, one can write these conditions as

$$\tfrac{3}{2} \geqslant L_{f'}(x), \tag{8}$$

$$3 \geqslant L_{f'}(x). \tag{9}$$

Here we have obtained for the case $f'(x) > 0$ the global convergence in terms of $L_{f'}(x)$. Our results agree with some particular cases in the more general study of Chebyshev's method given in [2,3].

References

[1] S. Amat, S. Busquier, J.M. Gutiérrez, Geometric constructions of iterative functions to solve nonlinear equations, J. Comput. Appl. Math. 157 (1) (2003) 197–205.

[2] M.A. Hernández, M.A. Salanova, A family of Chebyshev–Halley type methods, Internat. J. Comput. Math. 47 (1993) 59–63.

[3] M.A. Hernández, M.A. Salanova, Chebyshev method and convexity, Appl. Math. Comput. 95 (1998) 51–62.

[4] A. Melman, Geometry and convergence of Euler's and Halley's methods, SIAM Rev. 39 (4) (1997) 728–735.

[5] T.R. Scavo, J.B. Thoo, On the geometry of Halley's method, Amer. Math. Monthly 102 (1995) 417–426.

[6] J.F. Traub, Iterative Methods for Solution of Equations, Prentice-Hall, Englewood Cliffs, NJ, 1964.