A Unified Convergence Analysis for Some Two-Point Type Methods for Nonsmooth Operators

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Abstract: The aim of this paper is the approximation of nonlinear equations using iterative methods. We present a unified convergence analysis for some two-point type methods. This way we compare specializations of our method using not necessarily the same convergence criteria. We consider both semilocal and local analysis. In the first one, the hypotheses are imposed on the initial guess and in the second on the solution. The results can be applied for smooth and nonsmooth operators.

Keywords: iterative methods; nonlinear equations; Newton-type methods; smooth and nonsmooth operators

1. Introduction

One of the most important techniques in order to approximate nonlinear equations are iterative methods [1–6]. In this paper, we present a unified approach for two-point Newton-type methods for smooth and nonsmooth operators [7–10]. We will consider two types of convergence. The semilocal convergence is where the hypotheses are imposed on the initial guess; and local convergence is where the hypotheses are imposed on the solution. Our family includes a great variety of methods. We are interested also in the application of these methods in practice (nonlinear systems, boundary problems and image processing).

For a greater generality, in this study, let X and Y be two Banach spaces and D a nonempty, open, and convex set; let $F_1 : D \subset X \to Y$ and $F_2 : D \subset X \to Y$ be continuous operators. Moreover, we assume that the operator F_1 has a continuous Fréchet derivative and F_2 is a continuous operator whose differentiability is not assumed. We consider the equation

$$F(x) = F_1(x) + F_2(x) = 0. \tag{1}$$

To solve this equation, we use the two-point Newton-type methods defined by

$$x_{k+1} = x_k - L_{k-1,k}^{-1}(F_1(x_k) + F_2(x_k)) \tag{2}$$

for each $k = 0, 1, 2, \ldots$, where $x_{-1}, x_0 \in D$ are the initial points, $L(.,.) : D \times D \to \mathfrak{L}(X, Y)$ and $\mathfrak{L}(X, Y)$ is the space of bounded linear operators from X into Y. We have denoted by $L_{k-1,k} = L(x_{k-1}, x_k)$.

If $F_2(x) \neq 0$, we have that the operator F is not Fréchet differentiable. In general, to approximate a solution of (1) in this situation, derivative-free iterative methods are used [11–14]. To obtain this

type of iterative processes, it is common to approximate derivatives by difference divided. Remember that, given an operator $H : D \subset X \to Y$, we call $[x, y; H] \in \mathcal{L}(X, Y)$ a first order divided differences operator for H on the points x and y $(x \neq y)$ in D if

$$[x, y; H](x - y) = H(x) - H(y). \tag{3}$$

So, to solve (1) with iterative methods given from (2), we can consider at least two different procedures. Firstly, we have the Zincenko method [15], given by the following algorithm:

$$\begin{cases} \text{Given } x_{-1}, \ x_0 \in D, \\ x_{k+1} = x_k - \left[F_1'(x_k) \right]^{-1} F(x_k), \quad n \geq 0, \end{cases} \tag{4}$$

where we directly eliminate the nondifferentiable part of F, i.e., F_2. So, in this case, $L_{k-1,k} = F_1'(x_k)$ in (2). Secondly, we can consider an approximation of F' by divided differences, the secant-type methods [16,17]:

$$\begin{cases} \text{Given } x_{-1}, \ x_0 \in D, \\ y_k = \lambda x_k + (1 - \lambda) x_{k-1}, \quad \lambda \in [0, 1), \\ x_{k+1} = x_k - [y_k, x_k; F]^{-1} F(x_k), \quad n \geq 0, \end{cases} \tag{5}$$

where the secant method, for $\lambda = 0$, is obtained. So, in this case, $L_{k-1,k} = [y_k, x_k; F]$ in (2). But, if we consider a better approximation of the derivative of F, an approximation of second order, we have the Kurchatov method [18]:

$$\begin{cases} \text{Given } x_{-1}, \ x_0 \in D, \\ x_{k+1} = x_k - [x_{k-1}, 2x_k - x_{k-1}; F]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{6}$$

in this case, $L_{k-1,k} = [x_{k-1}, 2x_k - x_{k-1}; F]$ in (2).

By using this procedure of decomposition for operator F, we see that we can also consider the application of iterative methods that use derivatives when F is nondifferentiable. So, if we consider decomposition of F given in (1), we can use the Newton-secant-type algorithm:

$$\begin{cases} \text{Given } x_{-1}, \ x_0 \in D, \\ y_k = \lambda x_k + (1 - \lambda) x_{k-1}, \quad \lambda \in [0, 1), \\ x_{k+1} = x_k - \left(F_1'(x_k) + [y_k, x_k; F] \right)^{-1} F(x_k), \quad n \geq 0, \end{cases} \tag{7}$$

where $L_{k-1,k} = F_1'(x_k) + [y_k, x_k; F]$ in (2). The other possibility, from the decomposition method, is to consider the Newton–Kurchatov [19] algorithm:

$$\begin{cases} \text{Given } x_{-1}, \ x_0 \in D, \\ x_{k+1} = x_k - \left(F_1'(x_k) + [x_{k-1}, 2x_k - x_{k-1}; F] \right)^{-1} F(x_k), \quad n \geq 0, \end{cases} \tag{8}$$

where $L_{k-1,k} = F_1'(x_k) + [x_{k-1}, 2x_k - x_{k-1}; F]$ in (2). Another possibility is to consider Steffensen-type methods, that is, the methods associated to divided differences like $[x_k, x_k + F(x_k); F]$.

As we can see, there are a lot of iterative methods that can be written as algorithms (2).

The main aim of this paper is to obtain a general study for the convergence, local and semilocal, for these Newton-type of iterative methods given in (2).

2. Convergence Analysis for Two-Point Newton-Type Methods

In this section, we present both semilocal and local convergence analysis. In the first one, the hypotheses are imposed on the initial guess; and in the second, on the solution. The results can be applied for smooth and nonsmooth operators.

2.1. Local Convergence Analysis

We start the local analysis of method (2). Let $v_0 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function. Assume that the equation

$$v_0(t, t) = 1 \tag{9}$$

has at least one positive root r_0. Let also $v : [0, r_0) \times [0, r_0) \to [0, +\infty)$ be a nondecreasing continuous function. Define function $\bar{v}$ on the interval $[0, r_0)$ by

$$\bar{v}(t) = \frac{v(t, t)}{1 - v_0(t, t)} - 1. \tag{10}$$

Assume that the equation

$$\bar{v}(t) = 0 \tag{11}$$

has a minimal positive solution r. It follows that for each $t \in [0, r)$

$$0 \leq v_0(t, t) < 1 \tag{12}$$

and

$$0 \leq \bar{v}(t) < 1. \tag{13}$$

Our analysis of method (2) will use the conditions (A):

- (a_1) There exist a solution $x^* \in D$ of Equation (1), and $B \in \mathcal{L}(X, Y)$ such that $B^{-1} \in \mathcal{L}(Y, X)$.
- (a_2) Condition (9) holds and for each $x, u \in D$

$$\|B^{-1}(L(x, u) - B)\| \leq v_0(\|x - x^*\|, \|u - y^*\|),$$

  where v_0 is defined previously, and r_0 is given in (9).

  Set $D_0 = D \cap \bar{U}(x^*, r_0)$.
- (a_3) For each $x, z \in D_0$, and any solution $y \in D$ of Equation (1)

$$\|B^{-1}(F_1(x) + F_2(x) - L(z, x)y)\| \leq v(\|z - y\|, \|x - y\|)\|x - y\|,$$

  where v is defined previously, and $L(\cdot, \cdot) : D_0 \times D_0 \to \mathcal{L}(X, Y)$.
- (a_4) $\bar{U}(x^*, r) \subset D$, where r is given in (10).
- (a_5)

$$\xi := \frac{v(r, r)}{1 - v_0(r, r)} \in [0, 1).$$

We are able to perform our local analysis of method (2) based on the aforementioned conditions (A).

Theorem 1. *Assume that the conditions (A) hold. Then, sequence x_k, defined by method (2) for $x_{-1}, x_0 \in U(x^*, r) - x^*$, is well defined in $U(x^*, r)$; remains in $U(x^*, r)$; and converges to x^*. Finally, the following estimates hold.*

$$\|x_{k+1} - x^*\| \leq \frac{v(\|x_{k-1} - x^*\|, \|x_k - x^*\|)}{1 - v_0(\|x_{k-1} - x^*\|, \|x_k - x^*\|)} \|x_k - x^*\| \leq \|x_k - x^*\| < r. \tag{14}$$

The vector x^* is the only solution of Equation (1) in $\bar{U}(x^*, r)$.

Proof. We will use mathematical induction on k.

Let $x, u \in U(x^*, r)$.

Using (2), (a_1) and (a_2), we obtain

$$\|B^{-1}(L(x,u) - B)\| \le v_0(\|x - x^*\|, \|u - x^*\|) \le v_0(r, r) < 1. \tag{15}$$

Using the Banach lemma on invertible operators [20] and (15), we deduce that $L(x,u)^{-1} \in \mathfrak{L}(Y, X)$, and

$$\|L(x,u)^{-1}B\| \le \frac{1}{1 - v_0(\|x - x^*\|, \|u - x^*\|)}. \tag{16}$$

In particular, estimate (16) holds for $x = x_0$, so x_1 is well defined by method (2) for $k = 0$.

Using the definition of the method (2) (for $k = 0$); (a_1), (a_3), (13), and (16) (for $k = 0$) that

$$
\begin{aligned}
\|x_1 - x^*\| &= \|x_0 - x^* - L(x_{-1}, x_0)^{-1}(x_{-1}, x_0)(F_1(x_0) + F_2(x_0))\| \\
&= \|[L(x_{-1}, x_0)^{-1}B][B^{-1}(F_1(x_0) + F_2(x_0) - L(x_{-1}, x_0)(x_0 - x^*))]\| \\
&\le \|L(x_{-1}, x_0)^{-1}B\| \|B^{-1}(F_1(x_0) + F_2(x_0) - L(x_{-1}, x_0)(x_0 - x^*))\| \\
&\le \frac{v(\|x_{-1} - x^*\|, \|x_0 - x^*\|)}{1 - v_0(\|x_{-1} - x^*\|, \|x_0 - x^*\|)} \|x_0 - x^*\| \le \|x_0 - x^*\| < r,
\end{aligned}
\tag{17}
$$

which shows estimate (14) for $k = 0$ and $x_1 \in U(x^*, r)$.

Replace x_0, x_1 by x_i, x_{i+1} in the preceding estimates to complete the induction for estimate (14). Then, from the estimate

$$\|x_{i+1} - x^*\| \le \mu\|x_i - x^*\| < r, \tag{18}$$

where

$$\mu = \frac{v(\|x_{-1} - x\|, \|x_0 - x^*\|)}{1 - v_0(\|x_{-1} - x^*\|, \|x_0 - x^*\|)} \in [0, 1),$$

thus, $\lim_{i \to +\infty} x_i = x^*$ and $x_{i+1} \in U(x^*, r)$. Moreover, for the uniqueness part, let $y^* \in \bar{U}(x^*, r)$ with $F_1(y^*) + F_2(y^*) = 0$. Using $(a3)$, $(a5)$, and estimate (17), we obtain in turn that

$$
\begin{aligned}
\|x_{i+1} - y^*\| &\le \|L(x_{i-1}, x_i)^{-1}B\| \|B^{-1}(F_1(x_i) + F_2(x_i) - L(x_{i-1}, x_i)(x_i - y^*))\| \\
&\le \frac{v(\|x_{i-1} - y^*\|, \|x_i - y^*\|)}{1 - v_0(\|x_{i-1} - y^*\|, \|x_i - y^*\|)} \|x_i - y^*\| \\
&\le \mu\|x_i - y^*\| < \mu^{i+1}\|x_0 - y^*\|,
\end{aligned}
\tag{19}
$$

which shows $\lim_{i \to +\infty} x_i = y^*$—but, we showed $\lim_{i \to +\infty} x_i = x^*$. Hence, we conclude that $x^* = y^*$. $\square$

Remark 1. ● *Condition (a_3) can be replaced by the stronger: for each $x, y \in D_0$*

$$\|B^{-1}(F_1(x) + F_2(x) - L(x)(x - y))\| \le v_1(\|x - y\|)\|x - y\|,$$

where function v_1 is as v. However, for each $t, t' \ge 0$

$$v(t, t') \le v_1(t, t').$$

● *Linear operator B does not necessarily depend on q, where $q = x^*$ or $q = x_0$. It is used to determine the invertibility of linear operator $L(\cdot, \cdot)$ appearing in the method. The invertibility of B can be assured by*

an additional condition of the form $||I - B|| < 1$ or in some other way. A possible choice for B is $B = B(q)$ or $B = F_1'(q)$.

- *It follows from the definition of r_0 and r that $r_0 \geq r$.*

2.2. Semilocal Convergence Analysis

For the semilocal case, we also define some functions and parameters. Let $w_0 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous and nondecreasing function.

Assume that the equation

$$w_0(t, t) = 1, \tag{20}$$

has one smallest positive root that we denote by ρ_0. Let $w : [0, \rho_0) \times [0, \rho_0) \times [0, \rho_0) \to [0, +\infty)$ be a nondecreasing continuous function. Moreover, for $\eta, \bar{\eta} \geq 0$, define parameters C_1 and C_2 by

$$C_1 = \frac{w(\bar{\eta}, \eta, 0)}{1 - w_0(0, \eta)},$$

$$C_2 = \frac{w(0, \frac{\eta}{1 - C_1}, \eta)}{1 - w_0(\eta, \frac{\eta}{1 - C_1})}$$

and function $C : [0, \rho_0) \to [0, +\infty)$ by $C(t) = \frac{w(t,t,t)}{1 - w_0(t,t)}$. Assume that the equation

$$(\frac{C_1 C_2}{1 - C(t)} + C_1 + 1)\eta - t = 0 \tag{21}$$

has one smallest positive root that we denote by ρ.

The semilocal convergence analysis of method (2) will be based on conditions (H):

- ($h1$) There exists $x_{-1}, x_0 \in D$, and $B \in \mathfrak{L}(X, Y)$ such that $B^{-1} \in \mathfrak{L}(Y, X)$.
- ($h2$) Condition (20) holds, and for each $x \in D$

$$\|B^{-1}(L(z, x) - B)\| \leq w_0(\|z - x_0\|, \|x - x_0\|),$$

where w_0 is defined previously and ρ_0 is given in (20).

Set $D_1 = D \cap \bar{U}(x_0, \rho_0)$.
- ($h3$) For $L(\cdot, \cdot) : D_1 \times D_1 \to \mathfrak{L}(X, Y)$, and each $x, y, z \in D_1$

$$\|B^{-1}(F_1(y) - F_1(x) + F_2(y) - F_2(x) - L(z, x)(y - x))\|$$
$$\leq w(\|z - x_0\|, \|y - x_0\|, \|x - x_0\|)\|y - x\|,$$

where w is defined previously.
- ($h4$) $\bar{U}(x_0, \rho) \subseteq D$ and condition (21) holds for ρ, where $\|x_1 - x_0\| \leq \eta$ and $\|x_{-1} - x_0\| \leq \bar{\eta}$.

Then, using the hypotheses (H), we obtain the estimates:

$$
\begin{aligned}
\|x_2 - x_1\| &\leq \frac{w(\|x_{-1} - x_0\|, \|x_1 - x_0\|, \|x_0 - x_0\|)\|x_1 - x_0\|}{1 - w_0(\|x_0 - x_0\|, \|x_1 - x_0\|)} = C_1\|x_1 - x_0\|, \\
\|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (1 + C_1)\|x_1 - x_0\| \\
&= \frac{1 - C_1^2}{1 - C_1}\|x_1 - x_0\| \\
&< \frac{\eta}{1 - C_1} < \rho, \\
\|x_3 - x_2\| &\leq \frac{w(\|x_0 - x_0\|, \|x_2 - x_0\|, \|x_1 - x_0\|)}{1 - w_0(\|x_1 - x_0\|, \|x_2 - x_0\|)}\|x_2 - x_1\| \\
&\leq \frac{w(0, \frac{\eta}{1 - C_1}, \eta)}{1 - w_0(\eta, \frac{\eta}{1 - C_1})}\|x_2 - x_1\| = C_2\|x_2 - x_1\|, \\
\|x_3 - x_0\| &\leq \|x_3 - x_2\| + \|x_2 - x_1\| + \|x_1 - x_0\| \\
&\leq C_2\|x_2 - x_1\| + C_1\|x_1 - x_0\| + \|x_1 - x_0\| \\
&\leq (C_2 C_1 + C_1 + 1)\|x_1 - x_0\|, \\
\|x_4 - x_3\| &\leq \frac{w(\|x_1 - x_0\|, \|x_3 - x_0\|, \|x_2 - x_0\|)}{1 - w_0(\|x_2 - x_0\|, \|x_3 - x_0\|)}\|x_3 - x_2\| \\
&\leq C(\rho)\|x_3 - x_2\| \leq C(\rho)C_2\|x_2 - x_1\| \\
&\leq C(\rho)C_2 C_1\|x_1 - x_0\|,
\end{aligned}
\tag{22}
$$

similarly for $i = 3, 4, \ldots$

$$
\begin{aligned}
\|x_{i+1} - x_i\| &\leq C(\rho)\|x_i - x_{i-1}\| \leq C(\rho)^{i-2}\|x_3 - x_2\|, \\
\|x_{i+1} - x_0\| &\leq \|x_{i+1} - x_i\| + \ldots + \|x_4 - x_3\| + \|x_3 - x_0\| \\
&\leq C(\rho)\|x_i - x_{i-1}\| + \ldots + C(\rho)\|x_3 - x_2\| \\
&\quad + (C_2 C_1 + C_1 + 1)\|x_1 - x_0\| \\
&\leq C(\rho)^{i-2}\|x_3 - x_2\| + \ldots + C(\rho)\|x_3 - x_2\| \\
&\quad + (C_2 C_1 + C_1 + 1)\|x_1 - x_0\| \\
&\leq \left(\frac{1 - C(\rho)^{i-1}}{1 - C(\rho)}C_2 C_1 + C_1 + 1\right)\|x_1 - x_0\| \\
&< \left(\frac{C_1 C_2}{1 - C(\rho)} + C_1 + 1\right)\eta \leq \rho,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\|x_{i+j} - x_i\| &\leq \|x_{i+j} - x_{i+j-1}\| + \|x_{i+j-1} - x_{i+j-2}\| + \ldots + \|x_{i+1} - x_i\| \\
&\leq (C(\rho)^{i+j-3} + \ldots + C(\rho)^{i-2})\|x_3 - x_2\| \\
&\leq C(\rho)^{i-2}\frac{1 - C(\rho)^{j-1}}{1 - C(\rho)}\|x_3 - x_2\| \\
&\leq C(\rho)^{i-2}\frac{1 - C(\rho)^{j-1}}{1 - C(\rho)}C_2 C_1\|x_1 - x_0\| \\
&\leq C(\rho)^{i-2}\frac{1 - C(\rho)^{j-1}}{1 - C(\rho)}C_2 C_1 \eta.
\end{aligned}
\tag{24}
$$

It follows from (23) that $x_i \in U(x_0, \rho)$; and from (24) that sequence x_i is complete in a Banach space X. In particular, it converges to some $x^* \in \bar{U}(x_0, \rho)$. By letting $i \to +\infty$ in the estimate

$$\|B^{-1}(F_1(x_i) + F_2(x_i))\| = \|B^{-1}(F_1(x_i) + F_2(x_i) - F_1(x_{i-1}) - F_2(x_{i-1}) - L_{i-2,i-1}(x_i - x_{i-1}))\|$$

$$\leq \frac{w(\|x_{i-2} - x_0\|, \|x_i - x_0\|, \|x_{i-1} - x_0\|)\|x_i - x_{i-1}\|}{1 - w_0(\|x_{i-1} - x_0\|, \|x_i - x_0\|)} \leq \frac{w(\rho, \rho, \rho)}{1 - w_0(\rho, \rho)} \|x_i - x_{i-1}\|,$$

we obtain $F_1(x^*) + F_2(x^*) = 0$. The uniqueness part is omitted as analogous to the one in the local convergence case.

Hence, we can present our semilocal convergence result associated to the method (2).

Theorem 2. *Assume that the conditions (H) hold. Then, sequence x_k, defined by the method (2) for $x_{-1}, x_0 \in D$, is well defined in $U(x_0, \rho)$; remains in $U(x_0, \rho)$; and converges to a solution $x^* \in \bar{U}(x_0, \rho)$ of Equation (1). On the other hand, the vector x^* is the only solution of Equation (1) in $\bar{U}(x_0, \rho)$.*

The same comments given in the previous remark hold.

3. Numerical Experiment

Consider the nondifferentiable system of equations

$$\begin{cases} 3x_1^2 x_2 + x_2^2 - 1 + |x_1 - 1|^{3/2} = 0, \\ x_1^4 + x_1 x_2^3 - 1 + |x_2|^{3/2} = 0. \end{cases} \tag{25}$$

We therefore have an operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F = F_1 + F_2$, as in (1), with $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}^2$, $F_1 = (F_1{}^1, F_1{}^2)$, $F_2 = (F_2{}^1, F_2{}^2)$, being

$$F_1^1(x_1, x_2) = 3x_1^2 x_2 + x_2^2 - 1 \text{ and } F_1^2(x_1, x_2) = x_1^4 + x_1 x_2^3 - 1,$$

$$F_2^1(x_1, x_2) = |x_1 - 1|^{3/2} \text{ and } F_2^2(x_1, x_2) = |x_2|^{3/2},$$

where the operator F_1 is continuously Fréchet-differentiable and F_2 is continuous but is a Fréchet nondifferentiable operator.

For $\mathbf{u} = (u_1, u_2^T)$, $\mathbf{v} = (v_1, v_2)^T \in \mathbb{R}^2$, we consider the divided difference of first order defined by $[\mathbf{u}, \mathbf{v}; F] = ([\mathbf{u}, \mathbf{v}; F]_{ij})_{i,j=1}^2 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$, where

$$[\mathbf{u}, \mathbf{v}; F]_{i1} = \begin{cases} \dfrac{F_i(u_1, u_2, \ldots, v_m) - F_i(u_1, v_2, \ldots, v_m)}{u_1 - v_1}, & \text{if } u_2 \neq v_2, \\ 0, & \text{if } u_1 = v_1, \end{cases}$$

$$[\mathbf{u}, \mathbf{v}; F]_{i2} = \begin{cases} \dfrac{F_i(u_1, u_2, \ldots, v_m) - F_i(u_1, v_2, \ldots, v_m)}{u_2 - v_2}, & \text{if } u_2 \neq v_2, \\ 0, & \text{if } u_2 = v_2, \end{cases}$$

for $i = 1, 2$.

The iterative processes given by (1) allow us to consider direct iterative processes, such as (5) and (6); as well as iterative processes that use the decomposition method, such as (7) and (8). In this experiment, for the nondifferentiable system (25), we check that the application of the iterative processes that use the decomposition method have better behavior than the direct methods.

To carry out this study, we will consider as an approximate solution of system (25):

$$\mathbf{x}^* = (0.9383410452297656, 0.3312445136375143),$$

the starting points $\mathbf{x_{-1}} = (5, 5)$ and $\mathbf{x_0} = (1, 0)$, and use a tolerance $\|\mathbf{x_{n+1}} - \mathbf{x_n}\| \leq 10^{-16}$. In these conditions, in Tables 1 and 2 we can see the results of the application of the direct iterative processes, the secant-type, and Kurchatov methods. Whereas in Tables 3 and 4 we can see the results of the application of the iterative processes that use the decomposition method, Newton-secant-type and Newton–Kuchatov methods. Observing the results obtained, it is evident that the best behavior of the iterative processes is given by (2) using the decomposition method.

Table 1. $\|\mathbf{x}^* - \mathbf{x_n}\|$ for secant-type methods (5) and different values of the parameter λ.

n	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 0.99$
1	3.18484×10^{-1}	2.965×10^{-1}	4.54388×10^{-2}
2	5.21264×10^{-2}	4.13083×10^{-2}	3.74494×10^{-3}
3	3.66108×10^{-3}	2.35344×10^{-3}	2.94716×10^{-5}
4	2.59348×10^{-4}	7.2935×10^{-5}	3.69966×10^{-9}
5	1.30031×10^{-6}	1.24012×10^{-7}	2.10942×10^{-15}
6	4.42187×10^{-10}	6.07747×10^{-12}	
7	1.11022×10^{-15}	1.11022×10^{-16}	

Table 2. Kurchatov method (6).

n	$\|\mathbf{x}^* - \mathbf{x_n}\|$
1	2.99754×10^{-1}
2	1.07269×10^{-1}
3	4.20963×10^{-2}
4	8.37098×10^{-3}
5	2.78931×10^{-4}
6	9.0784×10^{-8}
7	2.9826×10^{-11}

Table 3. $\|\mathbf{x}^* - \mathbf{x_n}\|$ for Newton-secant-type methods (7) and different values of the parameter λ.

n	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 0.99$
1	2.3538×10^{-1}	1.00278×10^{-1}	4.29554×10^{-2}
2	3.48717×10^{-1}	2.88094×10^{-2}	2.53626×10^{-3}
3	1.47537×10^{-1}	1.90518×10^{-3}	9.06208×10^{-6}
4	3.4371×10^{-2}	8.39107×10^{-6}	1.9925×10^{-10}
5	3.08399×10^{-3}	4.78016×10^{-9}	
6	4.63665×10^{-5}	7.38298×10^{-15}	
7	4.05776×10^{-8}		
8	2.96929×10^{-13}		

Table 4. Newton–Kurchatov method (8).

n	$\|\overline{x}^* - \overline{x_n}\|$
1	6.1659×10^{-1}
2	4.75269×10^{-3}
3	6.29174×10^{-5}
4	3.37027×10^{-9}

Remark 2. *In the above example, we have selected the initial guess in a region where the operator is not smooth. The methods can be applied to systems where the operator is not smooth at the solution.*

For instance, for the system:

$$
\begin{cases}
|x_1^2 - 1| + x_2 - 1 = 0, \\
x_1 + x_2^2 - 1 = 0,
\end{cases}
\tag{26}
$$

the solution is $(1, 1)$. *If we take as initial guess* $(0.5, 0.5)$, *the Steffensen method gives as errors* 3.12×10^{-2}, 5.48×10^{-4}, 1.92×10^{-7}, 2.15×10^{-14}, *we observe its second order.*

4. Boundary Value Problem: Discretization via the Multiple Shooting Method

We will use the multiple shooting method for the discretization of boundary problems of the type

$$
y''(t) = f(t, y(t), y'(t)), \qquad y(a) = \alpha, \qquad y(b) = \beta.
\tag{27}
$$

Thus, we should find the solution of the following nonlinear system of equations $F(s) = 0$, where $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and

$$
\begin{cases}
F_1(s_0, s_1, \ldots, s_{N-1}) &=& s_1 - y'(t_1; s_0) \\
F_2(s_0, s_1, \ldots, s_{N-1}) &=& s_2 - y'(t_2; s_0, s_1) \\
&\vdots& \\
F_{N-1}(s_0, s_1, \ldots, s_{N-1}) &=& s_{N-1} - y'(t_{N-1}; s_0, s_1, \ldots, s_{N-2}) \\
F_N(s_0, s_1, \ldots, s_{N-1}) &=& \beta - y(t_N; s_0, s_1, s_{N-2}, s_{N-1}).
\end{cases}
$$

for a discretization of $[a, b]$ with N subintervals,

$$
T N j, \qquad T = b - a, \qquad j = 0, 1, \ldots, N.
$$

We consider the secant-type method

$$
\begin{cases}
\text{Given } y_{-1}, \ y_0 \in D, \\
z_n = \lambda_n y_n + (1 - \lambda_n) y_{n-1}, \quad \lambda_n \in [0, 1), \\
y_{n+1} = y_n - [z_n, y_n; F]^{-1} F(y_n), \quad n \geq 0,
\end{cases}
\tag{28}
$$

where λ_n is such that $||z_n - x_n|| \leq Tol$ for a given tolerance, and Newton's method

$$
\begin{cases}
\text{Given } y_0 \in D, \\
y_{n+1} = y_n - F'(y_n)^{-1} F(y_n), \quad n \geq 0.
\end{cases}
\tag{29}
$$

We perform a numerical comparison between both methods. As we can see, in the multiple shooting method, the iterative schemes are used as black boxes.

For the initial slope $\vec{s}_0 = (s_0^0, s_1^0, \ldots, s_{N-1}^0)$, we propose

$$
\begin{cases}
s_0^0 &= \dfrac{\beta - \alpha}{b - a} = \dfrac{y(t_N) - y(t_0)}{t_N - t_0}, \\[2mm]
s_1^0 &= \dfrac{y(t_N) - y(t_1; s_0)}{t_N - t_1}, \\[2mm]
s_2^0 &= \dfrac{y(t_N) - y(t_2; s_0, s_1)}{t_N - t_2}, \\[2mm]
&\vdots \\[2mm]
s_{N-1}^0 &= \dfrac{y(t_N) - y(t_{N-1}; s_0, s_1, \ldots, s_{N-2})}{t_N - t_{N-1}}.
\end{cases}
$$

We analyze this particular example ([21], p. 554):

$$
\begin{aligned}
y''(t) &= \tau \cdot \sinh(\tau \cdot y(t)), \\
y(0) &= 0, \quad y(1) = 1.
\end{aligned}
$$

We take $\tau = 2.5$ and $N = 4$ subintervals.

This problem admits the solution:

$$
y(t) = \frac{2}{\tau} \arg\sinh \left(\frac{s}{2} \cdot \frac{\operatorname{sn}\left(\tau t, 1 - s^2/4\right)}{\operatorname{cn}\left(\tau t, 1 - s^2/4\right)} \right),
$$

where

$$
s = y'(0) = 0.3713363932677645
$$

and $\operatorname{sn}(\cdot, \cdot)$ and $\operatorname{cn}(\cdot, \cdot)$ are the Jacobi elliptic functions.

Newton's method (29),

n	$\|F(\vec{s}_n)\|_\infty$	$\|y(t) - y_n\|_\infty$	$\|y'(t) - y_n'\|_\infty$
0	10^0	10^{-1}	10^0
1	10^{-1}	10^{-1}	10^{-1}
2	10^{-2}	10^{-2}	10^{-2}
3	10^{-4}	10^{-4}	10^{-4}
4	10^{-7}	10^{-7}	10^{-7}
5	10^{-15}	10^{-15}	10^{-14}

Secant-type method (28),

n	$\|F(\vec{s}_n)\|_\infty$	$\|y(t) - y_n\|_\infty$	$\|y'(t) - y_n'\|_\infty$	$\|F'(y_n) - [y_n, x_n; F]\|_\infty$
0	10^0	10^{-1}	10^0	10^{-6}
1	10^{-1}	10^{-1}	10^{-1}	10^{-6}
2	10^{-2}	10^{-2}	10^{-2}	10^{-7}
3	10^{-4}	10^{-4}	10^{-4}	10^{-6}
4	10^{-7}	10^{-7}	10^{-7}	10^{-6}
5	10^{-15}	10^{-15}	10^{-14}	10^{-6}

The methods using Jacobians obtain their order of convergence. However, in this example, the computation of the derivatives involves the approximation of a more complicated problem. For this reason, the methods free of derivatives are preferred, see [21]. Of course, we need to compute a good approximation to the Jacobian, this is the motivation of our parameters λ_n. For more similar examples and conclusions, we refer [22].

Remark 3. *In many cases, when we manipulate an image, some random noise appears. This noise makes the later steps of processing the image difficult and inaccurate.*

Let $f : \Omega \to \mathbb{R}$ be a noise signal or image.

Introducing the variable w:

$$w = \frac{\nabla u}{\sqrt{|\nabla u|^2}},$$

the Total-Variation model is equivalent to the nonlinear and nondifferentiable system:

$$
\begin{aligned}
-\nabla \cdot w + \lambda(u - f) &= 0, \\
w\sqrt{|\nabla u|^2} - \nabla u &= 0.
\end{aligned}
$$

This system should be discretized using finite differences and the associated nonlinear system of equations can be approximated by our family (see [23] for more details).

5. Conclusions

This paper was devoted to the analysis of a general family of two-point Newton-type methods for smooth and nonsmooth operators. We have considered two types of convergence—semilocal and local. The family includes a great number of methods. We have applied the schemes to several interesting problems, in particular to nonsmooth nonlinear systems, boundary problems, and image denoising models.

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