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Third-order iterative methods with applications to Hammerstein equations: A unified approach

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ABSTRACT

The geometrical interpretation of a family of higher order iterative methods for solving nonlinear scalar equations was presented in [S. Amat, S. Busquier, J.M. Gutiérrez, Geometric constructions of iterative functions to solve nonlinear equations. J. Comput. Appl. Math. 157(1) (2003) 197–205]. This family includes, as particular cases, some of the most famous third-order iterative methods: Chebyshev methods, Halley methods, super-Halley methods, C-methods and Newton-type two-step methods. The aim of the present paper is to analyze the convergence of this family for equations defined between two Banach spaces by using a technique developed in [J.A. Ezquerro, M.A. Hernández, Halley's method for operators with unbounded second derivative. Appl. Numer. Math. 57(3) (2007) 354–360]. This technique allows us to obtain a general semilocal convergence result for these methods, where the usual conditions on the second derivative are relaxed. On the other hand, the main practical difficulty related to the classical third-order iterative methods is the evaluation of bilinear operators, typically second-order Fréchet derivatives. However, in some cases, the second derivative is easy to evaluate. A clear example is provided by the approximation of Hammerstein equations, where it is diagonal by blocks. We finish the paper by applying our methods to some nonlinear integral equations of this type.

1. Introduction

This paper is devoted to the study of the convergence of some iterative methods for numerically approximating the solution of the nonlinear equations

$$F(x) = 0,$$

where F is a nonlinear function ranging from a Banach space X to another Banach space Y. We are interested in the following family of third-order iterative methods:

$$x_{n+1} = x_n - \left(I + \frac{1}{2}L_F(x_n)(I + b(x_n)F'(x_n)^{-1}F(x_n))^{-1}\right)F'(x_n)^{-1}F(x_n), \tag{1}$$

where I is the identity operator on X and for each $x \in X$, $L_F(x)$ is a linear operator on X defined by

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x),$$

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assuming that $F'(x)^{-1}$ exists. This family depends on the operator $b(x_n)$ that we have to make precise in order to define the above sequence in Banach spaces properly. In fact, $b(x_n)F'(x_n)^{-1}F(x_n)$ must be linear operators on X; a reasonable choice for the parameters $b(x_n)$ is then $b(x_n) = -F'(x_n)^{-1}B(x_n)$, with $B(x_n)$ any bilinear operator from $X \times X$ to Y.

This family was introduced for scalar equations $f(t) = 0$ in [1], after a geometrical interpretation of the most well-known third-order iterative methods. For instance, for different choices of the parameters $b(x_n)$, the following iterative schemes are included in the family (1):

1. *Halley's method*:
$$t_{n+1} = t_n - \left(\frac{1}{1+\frac{1}{2}L_f(t_n)}\right)\frac{f(t_n)}{f'(t_n)}.$$
2. *The super-Halley method*:
$$t_{n+1} = t_n - \left(1+\frac{L_f(t_n)}{2(1-L_f(t_n))}\right)\frac{f(t_n)}{f'(t_n)}.$$
3. *Chebyshev's method*:
$$t_{n+1} = t_n - \left(1+\frac{1}{2}L_f(t_n)\right)\frac{f(t_n)}{f'(t_n)}.$$
4. *C-methods*:
$$t_{n+1} = t_n - \left(1+\frac{1}{2}L_f(t_n)+CL_f(t_n)^2\right)\frac{f(t_n)}{f'(t_n)}, \quad 0 \le C \le 2.$$
5. *The two-step method*:
$$s_n = t_n - \frac{f(t_n)}{f'(t_n)}$$
$$t_{n+1} = s_n - \frac{f(s_n)}{f'(t_n)}.$$

A review of the amount of literature on high order iterative methods in the two last decades (see [2] and its references) may reveal the importance of these schemes. The main practical difficulty related to the classical third-order iterative methods is the evaluation of the second-order Fréchet derivative. For a nonlinear system of m equations and m unknowns, the first Fréchet derivative is a matrix with m^2 values, while the second Fréchet derivative has m^3 values. This implies a huge amount of operations in order to evaluate every iteration. Some methods overcome these difficulties by evaluating the function and its first derivative several times, such as the two-step method. This method is, in general, cheaper than any third-order methods requiring the evaluation of the second derivative. However, in some cases, the second derivative is easy to evaluate. A clear example is the approximation of Hammerstein equations. In other cases, we can choose the two-step method as a good alternative to the classical Newton method.

The rest of the paper is divided into two parts. In Section 2 we define a class of third-order methods and we analyze their convergence assuming different types of hypothesis [3–5]. In the third section, we present two examples of Hammerstein equations where we can apply the theory introduced. These equations are examples where the evaluation of the second derivative is not very expensive since it is diagonal by blocks.

2. Semilocal convergence

In this section, we analyze the semilocal convergence of the family (1) introduced in the previous section. In fact, we rewrite this family in the following form:

$$x_{n+1} = x_n - \left(I+\frac{1}{2}L_F(x_n)H(x_n)^{-1}\right)F'(x_n)^{-1}F(x_n), \tag{2}$$

where $H(x_n) = I - F'(x_n)^{-1}B(x_n)F'(x_n)^{-1}F(x_n)$ and $B(x_n)$ is a bilinear operator from $X \times X$ to Y to determine. Notice that a very natural choice for this operator is $B(x_n) = F''(x_n)$ but we can choose it depending on our interests.

Several techniques for finding sufficient conditions for the convergence of third-order processes can be found in the literature. Initially (see [6]), the strongest assumptions required for studying the convergence were $\|F''(x)\| \le M$ and $\|F'''(x)\| \le N$. Next (see [7–10]), the condition $\|F'''(x)\| \le N$ was replaced by the milder condition $\|F''(x)-F''(y)\| \le L\|x-y\|$ or by $\|F''(x) - F''(y)\| \le L\|x - y\|^p$, $p \in [0, 1]$, that is, F'' is Lipschitz continuous or (L, p)-Hölder continuous respectively. Finally, in [3], the convergence is obtained just assuming that

$$\|F''(x) - F''(y)\| \le \omega(\|x - y\|),$$

where ω is a positive non-decreasing continuous real function. Moreover, only the point condition $\|F''(x_0)\| \le M$ is assumed and not the stronger condition $\|F''(x)\| \le M$.

Following [3–5] and using an adaptation of their recurrence relations, we can obtain semilocal convergence results for our family of iterative methods. In this paper, we consider the most general case studied for instance in [5].

First, we remark that the family (2) is a particular case of the family

$$x_{n+1} = x_n - \mathcal{H}(\mathcal{L}_B(x_n), L_F(x_n))F'(x_n)^{-1}F(x_n), \tag{3}$$

with

$$\mathcal{H}(\mathcal{L}_B(x_n), L_F(x_n)) = I + \frac{1}{2}L_F(x_n) + L_F(x_n)\sum_{k\ge 2} A_k \mathcal{L}_B(x_n)^{k-1}, \quad \{A_k\}_{k\ge 2} \subset \mathbb{R}^+,$$

and

$$\mathcal{L}_B(x_n) = F'(x_n)^{-1}B(x_n)F'(x_n)^{-1}F(x_n).$$

We assume that the positive real sequence $\{A_k\}_{k\ge 2}$ is such that $\sum_{k\ge 2} A_k t^{k-1} < +\infty$ for $|t| < r$. If $\mathcal{L}_B(x_n)$ exists and $\|\mathcal{L}_B(x_n)\| < r$ then the method is well defined.

We assume the following hypotheses:

- H1 the operator $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists and $\|\Gamma_0\| \le \beta$,
- H2 $\|\Gamma_0 F(x_0)\| \le \eta$,
- H3 $\|F''(x_0)\| \le \alpha(F)$,
- H4 $\|B(x_0)\| \le \alpha(B)$,
- H5 $\|F''(x) - F''(y)\| \le \omega_F(\|x - y\|)$,
- H6 $\|B(x) - B(y)\| \le \omega_B(\|x - y\|)$,

where ω_F and ω_B are continuous non-decreasing functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\omega_F(0) = \omega_B(0) = 0$.

In order to improve the error estimate, we assume that a continuous and non-decreasing function ϕ from [0, 1] to [0, 1] exists, such that $\omega_F(\tau z) \le \phi(\tau)\omega_F(z)$ and $\omega_B(\tau z) \le \phi(\tau)\omega_B(z)$ for $\tau \in [0, 1]$. Observe that this is not a new restriction since we can always take $\phi(\tau) = 1$.

We define $L = \int_0^1 \phi(\tau)\mathrm{d}\tau$ and $M = \int_0^1 \phi(\tau)(1-\tau)\mathrm{d}\tau$.

From the initial conditions, we deduce

$$\begin{aligned}\|L_F(x_0)\| &\le \|\Gamma_0\|\,\|F''(x_0)\|\,\|\Gamma_0 F(x_0)\| \\ &\le \alpha(F)\beta\eta := a_0(F),\end{aligned}$$

$$\begin{aligned}\|\mathcal{L}_B(x_0)\| &\le \|\Gamma_0\|\,\|B(x_0)\|\,\|\Gamma_0 F(x_0)\| \\ &\le \alpha(B)\beta\eta := a_0(B).\end{aligned}$$

If $a_0(B) < r$, x_1 is well defined, since $\mathcal{H}(\mathcal{L}_B(x_0), L_F(x_0))$ exist and

$$\|\mathcal{H}(\mathcal{L}_B(x_0), L_F(x_0))\| \le 1 + \frac{1}{2}a_0(F) + a_0(F)\sum_{k\ge 2} A_k a_0(B)^{k-1}.$$

Hence

$$\begin{aligned}\|x_1 - x_0\| &\le \|\mathcal{H}(\mathcal{L}_B(x_0), L_F(x_0))\|\,\|\Gamma_0 F(x_0)\| \\ &\le \left(1 + \frac{1}{2}a_0(F) + a_0(F)\sum_{k\ge 2} A_k a_0(B)^{k-1}\right)\|\Gamma_0 F(x_0)\|.\end{aligned}$$

Throughout the paper we define:

- $a_0(F) = \alpha(F)\beta\eta$,
- $a_0(B) = \alpha(B)\beta\eta$,
- $b_0(B) = \beta\omega_B(h(a_0(B), a_0(F))\eta)\eta$,
- $b_0(F) = \beta\omega_F(h(a_0(B), a_0(F))\eta)\eta$,
- $c_0 = f(a_0(B), a_0(F), b_0(F))g(a_0(B), a_0(F), b_0(F))$,

where the following auxiliary real functions h, f and g are considered:

$$h(t, s) = 1 + \frac{1}{2}s + s\sum_{k\ge 2} A_k t^{k-1},$$

$$f(t, s, u) = \frac{1}{1 - h(t, s)(s + Lu)},$$

$$g(t, s, u) = h(t, s)\left(1 + h(t, s)\left(\frac{s}{2} + Mu\right)\right) - 1.$$

Next, we try to generalize the previous bounds to any step of iterative family (3). Then, we define the following auxiliary sequences:

- $a_n(B) = f(a_{n-1}(B), a_{n-1}(F), b_{n-1}(F))(b_{n-1}(B) + a_{n-1}(B))c_{n-1}$,
- $a_n(F) = f(a_{n-1}(B), a_{n-1}(F), b_{n-1}(F))(b_{n-1}(F) + a_{n-1}(F))c_{n-1}$,
- $b_n(B) = f(a_{n-1}(B), a_{n-1}(F), b_{n-1}(F))b_{n-1}(B)c_{n-1}\phi(c_{n-1})$,
- $b_n(F) = f(a_{n-1}(B), a_{n-1}(F), b_{n-1}(F))b_{n-1}(F)c_{n-1}\phi(c_{n-1})$,
- $c_n = f(a_n(B), a_n(F), b_n(F))g(a_n(B), a_n(F), b_n(F))$.

It is easy to check the following properties of the above sequences.

Lemma 1. (i) *If $h(a_0(B), a_0(F))(a_0(F) + Lb_0(F)) < 1$, $f(t, s, u)$ is increasing and $f(t, s, u) > 1$, for $t \in (0, a_0(B))$, $s \in (0, a_0(f))$ and $u \in (0, b_0(F))$.*
(ii) *$g(t, s, u)$ is increasing in all the components, with the others fixed.*

Lemma 2. *If $h(a_0(B), a_0(F))(a_0(F) + Lb_0(F)) < 1$, $a_1(B) < a_0(B)$ and $a_1(F) < a_0(F)$, then the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are decreasing. Besides, $c_0 < 1$.*

Lemma 3. *Let us suppose that $x_n \in \Omega$, for $n \in \mathbb{N}$. If the hypotheses of the above two lemmas are verified, then the following relations hold for all $n \geq 1$:*

- $[\mathrm{I}_n]\Gamma_n = F'(x_n)^{-1}$ *exists and* $\|\Gamma_n\| \leq f(a_{n-1}(B), a_{n-1}(F), b_{n-1}(F))\|\Gamma_n\|$,
- $[\mathrm{II}_n]\|\Gamma_n F(x_n)\| \leq c_{n-1}\|\Gamma_{n-1}F(x_{n-1})\| \leq c_0^n\|\Gamma_0 F(x_0)\|$,
- $[\mathrm{III}_n(B)]\|\Gamma_n\|\ \|B(x_n)\|\ \|\Gamma_n F(x_n)\| \leq a_n(B)$ *and* $\mathcal{H}(\mathcal{L}_B(x_n), L_F(x_n))$ *exists,*
- $[\mathrm{III}_n(F)]\|\Gamma_n\|\ \|F''(x_n)\|\ \|\Gamma_n F(x_n)\| \leq a_n(F)$,
- $[\mathrm{IV}_n]\|x_{n+1} - x_n\| \leq h(a_n(B), a_n(F))\|\Gamma_n F(x_n)\|$,
- $[\mathrm{V}_n]\|x_{n+1} - x_0\| \leq h(a_0(B), a_0(F))\sum_{k=0}^n c_0^k\|\Gamma_0 F(x_0)\|$,
- $[\mathrm{VI}_n(B)]\|\Gamma_n\|\omega_B(\|x_{n+1} - x_n\|)\|\Gamma_n F(x_n)\| \leq b_n(B)$,
- $[\mathrm{VI}_n(F)]\|\Gamma_n\|\omega_F(\|x_{n+1} - x_n\|)\|\Gamma_n F(x_n)\| \leq b_n(F)$.

Proof. We begin by proving that the conditions are satisfied for $n = 1$.

Firstly, we see that the inverse operator of $F'(x_1)$ exists. Since

$$F'(x_1) - F'(x_0) = \int_0^1 (F''(x_0 + t(x_1 - x_0)) - F''(x_0))(x_1 - x_0)\mathrm{d}t + \int_0^1 F''(x_0)(x_1 - x_0)\mathrm{d}t,$$

it follows that

$$\begin{aligned}\|F'(x_1) - F'(x_0)\| &\leq (L\omega_F(\|x_1 - x_0\|) + \alpha(F))\|x_1 - x_0\| \\ &\leq (L\omega_F(h(a_0(B), a_0(F))\eta) + \alpha(F))h(a_0(B), a_0(F))\eta.\end{aligned}$$

Hence

$$\begin{aligned}\|I - \Gamma_0 F'(x_1)\| &\leq \|\Gamma_0\|\ \|F'(x_1) - F'(x_0)\| \\ &\leq \beta(L\omega_F(h(a_0(B), a_0(F))\eta) + \alpha(F))h(a_0(B), a_0(F))\eta \\ &= (a_0(F) + Lb_0(F))h(a_0(B), a_0(F)) \\ &< 1.\end{aligned}$$

From Banach's lemma, Γ_1 exists and $[\mathrm{I}_1]$ follows, since

$$\begin{aligned}\|\Gamma_1\| &\leq \|\Gamma_1 F'(x_0)\|\ \|\Gamma_0\| \\ &\leq \frac{\|\Gamma_0\|}{1 - (a_0(F) + Lb_0(F))h(a_0(B), a_0(F))}.\end{aligned}$$

Now, we prove $[\mathrm{II}_1]$. For this, we use the following decomposition for the operator F, obtained from Taylor's formula:

$$F(x_1) = F(x_0) + F'(x_0)(x_1 - x_0) + \frac{1}{2}F''(x_0)(x_1 - x_0)^2 + \int_{x_0}^{x_1} (F''(x) - F''(x_0))(x_1 - x)\mathrm{d}x.$$

On the other hand, by the definition of the method

$$F'(x_0)(x_1 - x_0) = -F(x_0) - F''(x_0)\Gamma_0 F(x_0)\left(\frac{I}{2} + \sum_{k\geq 2} A_k L_B(x_0)^{k-1}\right)\Gamma_0 F(x_0).$$

Then, taking $x = x_0 + t(x_1 - x_0)$ in the Taylor expansion, it follows that

$$\begin{aligned} F(x_1) &= -F''(x_n)\Gamma_0 F(x_0)\left(\frac{I}{2} + \sum_{k\geq 2} A_k \mathcal{L}_B(x_0)^{k-1}\right)\Gamma_n F(x_0) + \frac{1}{2}F''(x_0)(x_1 - x_0)^2 \\ &\quad + \int_0^1 (F''(x_0 + t(x_1 - x_0)) - F''(x_0))(x_1 - x_0)^2(1-t)\mathrm{d}x \\ &= F''(x_0)\left(-\Gamma_0 F(x_0)\left(\frac{I}{2} + \sum_{k\geq 2} A_k \mathcal{L}_B(x_0)^{k-1}\right)\Gamma_0 F(x_0) + \frac{1}{2}(x_1 - x_0)^2\right) \\ &\quad + \int_0^1 (F''(x_0 + t(x_1 - x_0)) - F''(x_0))(x_1 - x_0)^2(1-t)\mathrm{d}x. \end{aligned}$$

Taking norms,

$$\begin{aligned} \|F(x_1)\| &\leq f(a_0(B), a_0(F), b_0(F))g(a_0(B), a_0(F), b_0(F))\|\Gamma_0 F(x_0)\| \\ &= c_0\|\Gamma_0 F(x_0)\| \end{aligned}$$

and $[\mathrm{II}_1]$ holds.

Moreover, from

$$\begin{aligned} \|B(x_1)\| &\leq \|B(x_1) - B(x_0)\| + \|B(x_0)\| \\ &\leq \omega_B(h(a_0(B), a_0(F))\eta) + \alpha(B), \end{aligned}$$

and

$$\begin{aligned} \|F''(x_1)\| &\leq \|F''(x_1) - F''(x_0)\| + \|F''(x_0)\| \\ &\leq \omega_F(h(a_0(B), a_0(F))\eta) + \alpha(F), \end{aligned}$$

we obtain $[\mathrm{III}_n(B)]$ and $[\mathrm{III}_n(B)]$ respectively, since

$$\begin{aligned} \|\Gamma_1\|\,\|B(x_1)\|\,\|\Gamma_1 F(x_1)\| &\leq f(a_0(B), a_0(F), b_0(F))\beta(\omega_B(h(a_0(B), a_0(F))\eta) + \alpha(B))c_0\eta \\ &\leq f(a_0(B), a_0(F), b_0(F))(b_0(B) + a_0(B))c_0 = a_1(B), \end{aligned}$$

and similarly

$$\|\Gamma_1\|\,\|F''(x_1)\|\,\|\Gamma_1 F(x_1)\| \leq f(a_0(B), a_0(F), b_0(F))(b_0(F) + a_0(F))c_0 = a_1(F).$$

Therefore $\|\mathcal{L}_B(x_1)\| \leq a_1(B)$ and since $\{a_n(B)\}$ is a decreasing sequence, $a_1(B) < r$. Then $(\mathcal{H}(\mathcal{L}_B(x_1), L_F(x_1)))$ exists and it follows that

$$\|(\mathcal{H}(\mathcal{L}_B(x_0), L_F(x_0)))\| \leq h(a_1(B), a_1(F)).$$

As a consequence of the above, $[\mathrm{IV}_1]$ is immediate:

$$\begin{aligned} \|x_2 - x_1\| &\leq \|(\mathcal{H}(\mathcal{L}_B(x_0), L_F(x_0)))\|\,\|\Gamma_1 F(x_1)\| \\ &\leq h(a_1(B), a_1(F))\|\Gamma_1 F(x_1)\|. \end{aligned}$$

Besides,

$$\begin{aligned} \|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\leq h(a_1(B), a_1(F))\|\Gamma_1 F(x_1)\| + h(a_0(B), a_0(F))\|\Gamma_0 F(x_0)\|, \end{aligned}$$

and taking into account that the real function h is increasing, $[\mathrm{V}_1]$ follows:

$$\|x_2 - x_0\| \leq h(a_0(B), a_0(F))(1 + c_0)\|\Gamma_0 F(x_0)\|.$$

Moreover, from

$$\begin{aligned} \|\Gamma_1\|\omega_B(\|x_2 - x_1\|)\|\Gamma_1 F(x_1)\| &\leq f(a_0(B), a_0(F), b_0(F))\beta\omega_B(h(a_1(B), a_1(F))c_0\eta)c_0\eta \\ &\leq f(a_0(B), a_0(F), b_0(F))b_0(B)c_0\phi(c_0) = b_1(B) \end{aligned}$$

and

$$\begin{aligned} \|\Gamma_1\|\omega_F(\|x_2 - x_1\|)\|\Gamma_1 F(x_1)\| &\leq f(a_0(B), a_0(F), b_0(F))\beta\omega_F(h(a_1(B), a_1(F))c_0\eta)c_0\eta \\ &\leq f(a_0(B), a_0(F), b_0(F))b_0(F)c_0\phi(c_0) = b_1(F), \end{aligned}$$

$[\mathrm{VI}_1(B)]$ and $[\mathrm{VI}_1(B)]$ hold.

After that, by applying mathematical induction, the proof is complete. □

Using this lemma it is easy to obtain the following semilocal result.

Theorem 1. *Let $F : \Omega \subseteq X \to Y$ be a nonlinear operator that is twice Fréchet differentiable on a non-empty open convex domain Ω. Suppose that the conditions $[\mathrm{I}_1]$–$[\mathrm{VI}_1(F)]$ are satisfied. Assume that $a_0 < r$, $a_1(B) < a_0(B)$, $a_1(F) < a_0(F)$ and $h(a_0(B), a_0(F))(a_0(F) + Lb_0(F)) < 1$. If $B(x_0, R) \subseteq \Omega$, where $R = \frac{h(a_0(B),a_0(F))\eta}{1-c_0}$, then the family of iterative processes given by (3), starting from x_0, converges to a solution x^* of $F(x) = 0$. In this case, $\{x_n\}$ and x^* belong to $\overline{B(x_0, R)}$.*

For the uniqueness we have:

Theorem 2. *Let us suppose that the hypotheses of Theorem 1 hold. Assume that there exists a positive root $\bar{R}$ of the equation*

$$\frac{\beta L}{\bar{R} - R} \int_R^{\bar{R}} \omega_F(u) u\mathrm{d}u + \beta\alpha(F)(R + \bar{R}) - 1 = 0.$$

Then, the solution x^* of $F(x) = 0$ is unique in $B(x_0, \bar{R}) \bigcap \Omega$.

We refer the reader to [5] for more details.

3. Approximation of the solution of Hammerstein equations

In many problems, an equation in the form

$$F(x) = 0,$$

where F is a nonlinear function ranging from a Banach space X to another, Y, has to be discretized in order to be solved computationally. The discretization process will produce a system of nonlinear scalar equations which increases its complexity as the discretization becomes finer. Thus, we approximate the solution of the original equation by means of the solution of the discretized one, x_n, verifying

$$F_n(x_n) = 0.$$

In this section, we are interested in pointing out a class of equations where the third-order methods studied in this paper are a good alternative to the Newton and two-step methods.

We shall consider an important special case of integral equations, known as the Hammerstein equations:

$$u(s) = \psi(s) + \int_0^1 H(s, t) f(t, u(t))\mathrm{d}t. \tag{4}$$

These equations are related to boundary value problems for differential equations. For some of them, third-order methods using second derivatives are useful for their effective (discretized) solution.

The discrete version of (4) is

$$x^i = \psi(t_i) + \sum_{j=0}^{m} \gamma_j H(t_i, t_j) f(t_j, x^j), \quad i = 0, 1, \ldots, m, \tag{5}$$

where $0 \leq t_0 < t_1 < \cdots < t_m \leq 1$ are the grid points of some quadrature formula $\int_0^1 f(t)\mathrm{d}t \approx \sum_{j=0}^m \gamma_j f(t_j)$, and $x^i = x(t_i)$.

Using the quadrature rule of integration in steps, we obtain a system of nonlinear equations. For simplicity, in this paper we consider the trapezoidal rule. The second Fréchet derivative of the associated discrete system is diagonal by blocks.

In order to illustrate our theoretical results numerically, let us consider the following two Hammerstein equations:

$$x(s) = 1 - \frac{1}{4} \int_0^1 \frac{s}{t + s} \frac{1}{x(t)} \mathrm{d}t, \quad s \in [0, 1] \tag{6}$$

and

$$x(s) = 1 + \int_0^1 G(s, t)(x(t)^{7/3} x(t)^3/3)\mathrm{d}t, \quad s \in [0, 1], \tag{7}$$

with the kernel

$$\begin{cases} (1-s)t, & t \leq s \\ s(1-t) & s \leq t. \end{cases} \tag{8}$$

These equations verify the hypothesis of Theorem 1. Notice that the first one verifies the classical conditions on the second derivative, that is $\|F''(x)\| \leq M$ and $\|F''(x) - F''(y)\| \leq L\|x - y\|$ (see [2]), but the second one only verifies the milder conditions $\|F''(x_0)\| \leq M$ and

$$\|F''(x) - F''(y)\| \leq \omega(\|x - y\|),$$

where ω is a positive non-decreasing continuous real function (see [3]).

Table 1
Errors for the methods (9)–(12), applied to Eq. (6) with $x_0 = 1.5$.

Iter.	Chebyshev	Halley	Two-step	$C = \frac{1}{2}$
1	$1.8\ 10^{-2}$	$1.7\ 10^{-2}$	$4.2\ 10^{-3}$	$1.7\ 10^{-2}$
2	$1.2\ 10^{-6}$	$1.0\ 10^{-6}$	$4.0\ 10^{-9}$	$8.2\ 10^{-7}$
3	0	0	0	0

We consider $m = 20$ in the quadrature trapezoidal formula and as the exact solution that obtained numerically by the Newton method. In Tables 1 and 3, we summarize the numerical results for different methods in the family (2):

- Chebyshev's method, obtained for $B(x_n) = 0$ and $H(x_n) = I$:

$$x_{n+1} = x_n - \left(I + \frac{1}{2}L_F(x_n)\right) F'(x_n)^{-1}F(x_n). \tag{9}$$

- Halley's method, obtained for $B(x_n) = -1/2F''(x_n)$ and $H(x_n) = I + (1/2)L_F(x_n)$:

$$x_{n+1} = x_n - \left(I + \frac{1}{2}L_F(x_n)[I + (1/2)L_F(x_n)]^{-1}\right) F'(x_n)^{-1}F(x_n). \tag{10}$$

- The 1/2-method, obtained for $B(x_n) = F''(x_n)[I + L_F(x_n)]^{-1}$ and $H(x_n) = [I + L_F(x_n)]^{-1}$:

$$x_{n+1} = x_n - \left(I + \frac{1}{2}L_F(x_n) + \frac{1}{2}L_F(x_n)\right) F'(x_n)^{-1}F(x_n). \tag{11}$$

- The two-step method, defined by

$$\begin{aligned} y_{n+1} &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_{n+1} - F'(x_n)^{-1}F(y_{n+1}). \end{aligned} \tag{12}$$

  and obtained in (2) if we consider $B(x_n)$ the bilinear operator from $X \times X$ to Y such that

$$\frac{1}{2}L_F(x_n)H(x_n)^{-1}F'(x_n)^{-1}F(x_n) = F'(x_n)^{-1}F(y_{n+1}).$$

For a comparison, we have selected four iterative methods M_i without involving second derivatives, introduced in [11–13].

- M_1 [11]:

$$\begin{aligned} y_{n+1} &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_{n+1} - \frac{1}{2}F'(x_n)^{-1}\left(F'(x_n) - F'(y_{n+1})\right) F'(x_n)^{-1}F(x_n). \end{aligned} \tag{13}$$

- M_2 [12]:

$$\begin{aligned} y_{n+1} &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \frac{1}{2}\left(F'(x_n)^{-1} + \left(2F'\left(\frac{1}{2}(x_n + y_{n+1})\right) - F'(x_n)\right)^{-1}\right) F(x_n). \end{aligned} \tag{14}$$

- M_3 [13]:

$$\begin{aligned} y_{n+1} &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \frac{1}{2}\left(F'(x_n)^{-1} + F'(y_{n+1})^{-1}\right) F(x_n). \end{aligned} \tag{15}$$

- M_4 [13]:

$$\begin{aligned} y_{n+1} &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \frac{1}{2}\left(F'(x_n) + F'(y_{n+1})\right)^{-1} F(x_n). \end{aligned} \tag{16}$$

The stopping criterion that we consider is $\|F(x_n)\| \le 10^{-16}$.

In Tables 1–4 we show the errors (working with the max-norm) of the four aforementioned methods selected from the family (2) and four other methods that do not belong to this family.

Table 2
Errors for the methods (13)–(16), applied to Eq. (6) with $x_0 = 1.5$.

Iter.	M_1	M_2	M_3	M_4
1	$2.1\ 10^{-2}$	$3.7\ 10^{-2}$	$7.2\ 10^{-3}$	$2.7\ 10^{-2}$
2	$2.3\ 10^{-6}$	$2.1\ 10^{-6}$	$1.2\ 10^{-8}$	$1.2\ 10^{-6}$
3	0	0	0	0

Table 3
Errors for the methods (9)–(12), applied to Eq. (7) with $x_0 = 0$.

Iter.	Chebyshev	Halley	Two-step	$C = \frac{1}{2}$
1	$2.1\ 10^{-2}$	$1.4\ 10^{-2}$	$5.2\ 10^{-2}$	$1.6\ 10^{-2}$
2	$1.8\ 10^{-6}$	$1.1\ 10^{-6}$	$7.8\ 10^{-7}$	$1.2\ 10^{-6}$
3	0	0	0	0

Table 4
Errors for the methods (13)–(16), applied to Eq. (7) with $x_0 = 0$.

Iter.	M_1	M_2	M_3	M_4
1	$5.1\ 10^{-2}$	$2.7\ 10^{-2}$	$4.3\ 10^{-2}$	$4.7\ 10^{-2}$
2	$1.3\ 10^{-6}$	$2.3\ 10^{-6}$	$3.2\ 10^{-6}$	$2.2\ 10^{-6}$
3	0	0	0	0

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